

Affine group representation formalism for four dimensional, Lorentzian, quantum gravity

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The Hamiltonian constraint of 4-dimensional General Relativity is recast explicitly in terms of the Chern–Simons functional and the local volume operator. In conjunction with the algebraic quantization program, application of the affine quantization concept due to Klauder facilitates the construction of solutions to all of the quantum constraints in the Ashtekar variables and their associated Hilbert space. A physical Hilbert space is constructed for Lorentzian signature gravity with nonzero cosmological constant in the form of unitary, irreducible representations of the affine group.

INTRODUCTION

Consistent quantization of gravity has remained one of the most challenging problems in theoretical physics. To meet the standard criteria for quantization via the canonical approach, one attempts to construct observables and to define a Hilbert space structure consistent with all of the constraints of the theory. Many aspects of quantization regarding the kinematic constraints of gravity have been addressed via the loop quantum approach (see, for instance, [1] and references therein). In Lorentzian loop quantum gravity (LQG) one defines a kinematic Hilbert space \mathbf{H}_{Kin} using the spin network states as a basis. These are constructed from holonomies over one dimensional graphs embedded in three space Σ , known as edges, and the edges are contracted with intertwiners to make the states $SU(2)$ invariant. The spin network states are eigenstates of volume and area, but are not in general annihilated by the Hamiltonian and the diffeomorphism constraints. To address the diffeomorphism constraint one performs a group averaging procedure to the spin network states over the diffeomorphism group. Since these states form a unitary representation of spatial diffeomorphisms, then one ends up with a diffeomorphism invariant, rigged, Hilbert space \mathbf{H}_{diff} . The construction of physical states of quantum gravity, namely states within \mathbf{H}_{diff} solving the Hamiltonian constraint, remains an unresolved problem in this approach[19].

There is one known exact solution to all of the constraints of vacuum General Relativity, the Chern-Simons state ψ_{CS} , which as well exhibits a good semiclassical limit [2], [3]. While lying in the simultaneous kernel of all of the constraints for a particular operator ordering, this solution may not meet the rigorous definition of a state owing to issues of normalizability and unitarity raised by Witten and others [4]. In the present paper we will reveal the existence of additional solutions to the constraints of Lorentzian gravity and within which context these solutions form a genuine physical Hilbert space \mathbf{H}_{Phys} . Some of the motivation for this work comes from the results of [5], where one can rewrite the Hamiltonian constraint in a vastly simplified form using geometric invariants. The main quantities of interest for the present paper will be the imaginary part of the same Chern-Simons functional which defines ψ_{CS} , and the local and global volume operators.

There is another result in quantum gravity which will be important for this paper, namely the affine quantization program started by Klauder [6]. In this approach, which was originally developed in the metric representation, the spatial 3-metric q_{ij} must satisfy certain positivity requirements. These requirements appear to be best implemented, in the quantum theory, via the affine group. In this paper we will construct some new solutions to the constraints of Lorentzian gravity, using the affine group within the context of the Ashtekar variables. We will show that these solutions exhibit a natural Hilbert space structure and can be classified as physical states for gravity.

The organization of this paper is as follows. In section 2 we provide some background on the Ashtekar variables and establish the notations to set the stage for this paper. Section 3 recapitulates some results of [5], expressing the Hamiltonian constraint using Poisson brackets involving the Chern-Simons functional $I_{CS}[A]$ and the local volume $V(x)$. Section 4 reveals the remarkable relation between the affine group and quantum gravity with Ashtekar variables; the quantization program is carried out using the affine group in conjunction with relevant steps of the algebraic quantization program[7], and a physical Hilbert space for Lorentzian gravity is constructed. We end with a brief summary of our results.

ASHTEKAR VARIABLES

Let M be a four dimensional spacetime manifold of topology $M = \Sigma \times R$, where Σ is a spatial 3-manifold of a given topology embedded in M . Then define on each Σ the canonical pair (A_i^a, \tilde{E}_a^i) , where A_i^a a $SO(3)$ gauge potential, and \tilde{E}_a^i a densitized triad of density weight 1 constructed from undensitized spatial triads e_i^a (index conventions are that symbols from the beginning of the Latin alphabet a, b, c, \dots label internal $SU(2)$ indices, while from the middle i, j, k, \dots label spatial indices). These are given by

$$A_i^a = \Gamma_i^a + \gamma K_i^a, \quad \tilde{E}_a^i = \frac{1}{2} \tilde{\epsilon}^{ijk} \epsilon_{abc} e_j^b e_k^c; \quad (1)$$

where γ is the Barbero-Immirzi parameter, Γ_i^a is the spin connection compatible with e_i^a , and K_i^a is the triadic form of the extrinsic curvature of Σ . Then the action for four dimensional gravity in the Ashtekar variables can be written in 3+1 form as [8–10]

$$I = \int dt \int_{\Sigma} d^3x \left(\frac{1}{\gamma G} [\tilde{E}_a^i \dot{A}_i^a + A_0^a G_a + N^i H_i] + N \mathcal{H} \right). \quad (2)$$

The equations of motion for the auxiliary fields A_0^a, N^i, N imply the vanishing of the Gauss' law, vector and Hamiltonian constraints respectively G_a, H_i and \mathcal{H} given by

$$\begin{aligned} G_a &= D_i \tilde{E}_a^i = \partial_i \tilde{E}_a^i + \epsilon_{ab}^c A_i^b \tilde{E}_c^i, \\ H_i &= \tilde{E}_a^j F_{ij}^a - \frac{1+\gamma^2}{\gamma} K_i^a G_a, \\ \mathcal{H} &= \frac{1}{2G\sqrt{|\det q|}} \tilde{E}_a^i \tilde{E}_b^j \left(\epsilon_c^{ab} F_{ij}^c + \frac{\Lambda}{3} \epsilon^{abc} \epsilon_{ijk} \tilde{E}_c^k - 2(1+\gamma^2) K_{[i}^a K_{j]}^b \right) + \frac{(1+\gamma^2)}{G\gamma^2} \left(\frac{\tilde{E}_a^i}{\sqrt{|\det q|}} \right) \partial_i G^a. \end{aligned} \quad (3)$$

Equation (2) provides the fundamental nontrivial Poisson bracket (in the action and elsewhere we have made the identification $\kappa = G$ to avoid carrying numerical factors around)

$$\{A_i^a(x), \tilde{E}_b^j(y)\} = \gamma G \delta_b^a \delta_i^j \delta^{(3)}(x, y), \quad (4)$$

which in conjunction with the reality conditions constitutes a basis for computation of the Hamiltonian dynamics of four dimensional General Relativity and its quantization.

For real values of γ the connection A_i^a is real and one also has gravity of Lorentzian signature where the need to implement reality conditions has been eliminated. On the other hand, the Hamiltonian constraint in this case is difficult to implement on account of the presence of extrinsic curvature squared and other terms involving γ . Nevertheless, this is the standard case utilized for the quantization of LQG. For $\gamma = \pm i$ the aforementioned offending terms vanish and the connection A_i^a becomes complex. While this yields a simple, polynomial Hamiltonian constraint, one must impose reality conditions in order to get real General Relativity of Lorentzian signature.

After having defined the theory at the classical level, the next natural step is to perform a quantization. There is no unique prescription for the extrapolation from classical to quantum theory. A standard approach to canonical quantization proceeds according to the Heisenberg prescription, wherein one promotes the dynamical variables to quantum operators $A_i^a, \tilde{E}_a^i \rightarrow \hat{A}_i^a, \hat{\tilde{E}}_a^i$ and defines a set of unit vectors $|\psi\rangle \in \mathbf{H}_{Kin}$ on which the operators act. These state vectors form a kinematic Hilbert space \mathbf{H}_{Kin} , which is the Hilbert space at the level prior to the implementation of any constraints. All Poisson brackets would become promoted to $\frac{1}{i\hbar}$ times quantum commutators, and so equation (4) would become (for $\gamma = i$) promoted to equal time commutation relations

$$[\hat{A}_i^a(x, t), \hat{\tilde{E}}_b^j(y, t)] = -(\hbar G) \delta_b^a \delta_i^j \delta^{(3)}(x, y). \quad (5)$$

The initial value constraints would become promoted to operator constraints \hat{G}_a, \hat{H}_i and \hat{H} for a prescribed operator ordering. According to the Dirac procedure for constrained systems [11], the physical states $\psi \in \mathbf{H}_{Phys}$ are those elements of \mathbf{H}_{Kin} which are annihilated by all of the quantum constraints

$$\hat{G}_a(x)|\psi\rangle = \hat{H}_i(x)|\psi\rangle = \hat{H}(x)|\psi\rangle = 0. \quad (6)$$

This amounts to finding a set of gauge-invariant, diffeomorphism-invariant functionals lying in the kernel of the quantum Hamiltonian constraint, which admits a Hilbert space structure.

In the LQG approach, one does not quantize all of the classical degrees of freedom in the canonical structure of the action (2). Rather, one starts with a different set of Poisson brackets based on certain extended objects classically constructible from these degrees of freedom, given by

$$\begin{aligned} U[\gamma, A] &= Tr \left[\hat{P} e^{\int_\gamma A} \right]; \\ E_a(S) &= \int_S d^2 x n_i \tilde{E}_a^i. \end{aligned} \quad (7)$$

Equation (7) consists of traces of holonomies around closed loops γ with path-ordering operator \hat{P} , and electric fluxes through 2-surfaces with normal vector n_i . These have a closed algebra under Poisson brackets [1]

$$\{U[\gamma, A], E_a(S)\} = Int[\gamma, S] U[\gamma_1, A] \tau_a U[\gamma_2, A], \quad (8)$$

where $Int[\gamma, S]$ is the intersection number of S and γ , γ_1 is the path from the origin to the point of intersection, and γ_2 is the rest of the path. It is then (8), in lieu of (4), which becomes promoted to commutators in the quantum theory. As outlined in the introduction, one can obtain a gauge invariant, diffeomorphism invariant Hilbert space using the standard techniques of LQG. However, the construction of a physical Hilbert space is presently an unresolved

problem due to the dilemma presented by the tradeoff between aforementioned reality conditions involving γ , and the tractability of the Hamiltonian constraint.

In the present paper we will provide a possible resolution via a different approach, which shares a particular feature in common with LQG. We will construct a physical Hilbert space using a set of suitable geometric extended objects as the starting point for Poisson brackets. But these objects will not be defined on graphs and surfaces as in (7). Rather, they will be defined on 3-space Σ . This will entail a reformulation of the Hamiltonian constraint and states explicitly and completely in terms of these objects. From now on, we will restrict consideration to the (anti)self-dual case of $\gamma = \mp i$ (for concreteness we adopt $\gamma = i$), which will constitute the fundamental basis for the results of this paper.

REFORMULATION OF THE HAMILTONIAN CONSTRAINT

From the spatial curvature F_{ij}^a let us define the Ashtekar magnetic field of the connection A_i^a , an object of density weight one given by

$$\tilde{B}^{ia} = \frac{1}{2}\epsilon^{ijk}F_{jk}^a. \quad (9)$$

When the densitized lapse function $\underline{N} = N/\sqrt{|\det q|}$ is treated as a fundamental auxiliary field, then the Hamiltonian constraint is a polynomial constraint of density weight two, given (we have omitted the double-tilde on H for simplicity, since it should be clear from the context) by

$$\frac{\delta I}{\delta \underline{N}(x)} = 0 \Leftrightarrow H(x) = \epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i\tilde{E}_b^j\tilde{B}_c^k + \frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i\tilde{E}_b^j\tilde{E}_c^k = 0. \quad (10)$$

One of the steps in [5] in reformulation of the Hamiltonian constraint is that the Chern–Simons functional $I_{CS}[A]$ of the connection A_i^a has the Poisson bracket

$$\{I_{CS}[A], \tilde{E}_c^k(x)\} = \int_{\Sigma} d^3y \tilde{B}_b^j(y) \{A_j^b(y), \tilde{E}_c^k(x)\} = iG\tilde{B}_c^k(x). \quad (11)$$

The curvature term of (10) can thus be written using this Poisson bracket by contracting (11) with two factors of \tilde{E}_a^i in antisymmetric combination and using the definition of determinants, yielding

$$\{I_{CS}[A], \det \tilde{E}(x)\} = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i(x)\tilde{E}_b^j(x)\{I_{CS}[A], \tilde{E}_c^k(x)\} = \frac{iG}{2}\epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i(x)\tilde{E}_b^j(x)\tilde{B}_c^k(x).$$

Let us define a local volume squared operator $V^2(x)$, given by

$$V^2(x) = \frac{1}{6}\epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i(x)\tilde{E}_b^j(x)\tilde{E}_c^k(x) = \det \tilde{E}(x). \quad (12)$$

Then using (12), equation (10) can be written in the following form

$$\epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i\tilde{E}_b^j\tilde{B}_c^k(x) + 2\Lambda\det \tilde{E}(x) = -\frac{2i}{G}\{I_{CS}[A], V^2(x)\} + 2\Lambda V^2(x) = 0. \quad (13)$$

Equation (13) is a polynomial constraint, a highly desirable feature from the standpoint of the quantum theory. Indeed, one can define the Wheeler–DeWitt equation, with symmetric factor-ordering, from the fundamental commutation relations (5) (omitting the hats for simplicity) by [5]

$$\begin{aligned} -\frac{2}{\hbar G}[I_{CS}[A], V^2(x)] + 2\Lambda V^2(x) &= \frac{1}{3}\epsilon_{ijk}\epsilon_{abc}(\tilde{E}^{ia}\tilde{E}^{jb}\tilde{B}^{kc} + \tilde{E}^{ia}\tilde{B}^{jb}\tilde{E}^{kc} + \tilde{B}^{ia}\tilde{E}^{jb}\tilde{E}^{kc}) + \frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\tilde{E}_a^i\tilde{E}_b^j\tilde{E}_c^k \\ &=: H = 0. \end{aligned} \quad (14)$$

Equation (14) is also the Weyl ordering of \tilde{E} and \tilde{B} . Moreover, the constraints algebra for a symmetric ordering of the constraints is closed, which is necessary for consistency in four dimensional quantum gravity. Thus this commutator has a useful role. It has been pointed out in [12] that background independent field theories are ultraviolet self-regulating if the constraint weight is equal to one but not for other density weights. This suggests that a more appropriate form for the Hamiltonian constraint for quantization is as a density weight one constraint. A density

weight one Hamiltonian constraint can be accomplished by treating the lapse N rather than the densitized lapse \underline{N} as the basic auxiliary field in the action (2). So vis-a-vis (10), variation of the lapse $N(x)$ yields

$$[\det \tilde{E}(x)]^{-\frac{1}{2}} (\epsilon_{ijk} \epsilon^{abc} \tilde{E}_a^i \tilde{E}_b^j \tilde{E}_c^k + \frac{\Lambda}{3} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_a^i \tilde{E}_b^j \tilde{E}_c^k) = -\frac{2i}{G} [\det \tilde{E}(x)]^{-\frac{1}{2}} (\{I_{CS}[A], \det \tilde{E}(x)\} + iG\Lambda \det \tilde{E}(x)) = 0. \quad (15)$$

For reasons related to diffeomorphism invariance which will come up later, we will need to write the Hamiltonian constraint in terms of the square root of $\det \tilde{E}(x)$. From the chain rule, the Poisson bracket

$$\{I_{CS}[A], \sqrt{|\det \tilde{E}(x)|}\} = \frac{1}{2} (|\det \tilde{E}(x)|)^{-1/2} \{I_{CS}[A], \det \tilde{E}(x)\} \quad (16)$$

holds, and substitution of (16) into (15) yields the weight one Hamiltonian constraint as

$$\{I_{CS}[A], \sqrt{|\det \tilde{E}(x)|}\} = -\frac{iG\Lambda}{2} \sqrt{|\det \tilde{E}(x)|}. \quad (17)$$

Equation (17) is however nonpolynomial in terms of \tilde{E}_a^i on account of the presence of the square root, which is an undesirable feature from the standpoint of quantization. To address this let us define the local volume operator

$$V(x) = \sqrt{\left| \frac{1}{6} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_a^i(x) \tilde{E}_b^j(x) \tilde{E}_c^k(x) \right|}, \quad (18)$$

the absolute value sign put in so that $V(x)$ is real both for positive and negative triad orientations (while $V(x)$ is of density weight one, we will omit the tilde symbol over it for notational simplicity. This should not lead to any confusion, and the proper density weight should be understood from the context). Note that in terms of $V(x)$, the preferred local density weight one Hamiltonian constraint can now be written in the polynomial form (in V and A) as

$$\{I_{CS}[A], V(x)\} = -\frac{iG\Lambda}{2} V(x) \quad \forall x. \quad (19)$$

Arguments supporting the notion that (19) should be rigorously true also at the quantum level in congruity with (14) will be provided.

AFFINE GROUP AND QUANTUM GRAVITY WITH ASHTEKAR VARIABLES

The importance of the affine group for quantum gravity was first pointed out by Klauder in [6], [13], [14], and the general concepts of continuous representation theory for the affine group has been developed by Klauder and Aslaksen in [15]. It is well known, from the Stone–Von Neuman theorem, that there is only one irreducible representation up to unitary equivalence of canonical, self-adjoint operators p and q satisfying the Weyl form of the canonical commutation relations $[q, p] = -i$. This representation, equivalent to the Schrödinger representation, implies that the spectrum of both p and q cover the whole real line. The affine commutation relation for a single degree of freedom takes the form [15]

$$[D, q] = -iq, \quad (20)$$

where $D = (pq + qp)/2$ denotes the dilation operator and q is the operator being dilated. It has been established that there exist two and only two unitarily inequivalent, irreducible representations π of the affine group G_{aff} , one representation π^+ for which the spectrum of q is positive and another π^- for which it is negative. This provided the motivation for a multidimensional generalization of the affine group provided by Klauder, where the dilated objects become replaced with operators corresponding to the spatial 3-metric q_{ij} . As noted in [6], the metric must satisfy certain positivity requirements which must be reflected in the quantum theory. Appendix A is a supplement on the measure, inner product, and coherent states associated with the affine group.

A comparison of (19) with (20) reveals that the local Hamiltonian constraint (19) is nothing other than the Lie algebra of affine transformations of the straight line. More precisely, it is an infinite number affine Lie algebras, one Lie algebra $g_{aff}(x)$ per spatial point x . The Chern–Simons functional $I_{CS}[A]$ plays the role of the dilator, and $V(x)$ plays the role of the object being dilated. By applying the affine quantum gravity concept to the Ashtekar variables, we will ultimately construct physical Hilbert spaces based upon the representation π^+ , thereby endowing the local volume operator for $V(x)$ with a positive spectrum. (The treatment of both π^+ and π^- would be tantamount to considering the two different orientations of the densitized triad \tilde{E}_a^i . This allows the possibility of topology change

in Lorentzian quantum gravity, where the quantum theories in each topological sector would be inequivalent. By restricting attention to $V(x) \neq 0$ we eliminate this possibility). The important caveat is that I_{CS} is not Hermitian due to complex self-dual Ashtekar variables, but we shall show that in (19) only the imaginary part of the Chern-Simons functional, Q , is relevant and thus the affine representation correspondence can be made exact.

We have shown the local density weight one Hamiltonian constraint $H(x) = 0$, as shown in (19) can classically be written as a Poisson bracket

$$\{-iI_{CS}[A], V(x)\} = -\left(\frac{G\Lambda}{2}\right)V(x) \quad \forall x. \quad (21)$$

Let us make the following definitions for the real and the imaginary parts of the Chern-Simons functional

$$Y = \text{Re}[I_{CS}[A]]; \quad Q = \text{Im}[I_{CS}[A]]. \quad (22)$$

Then at the classical level, the Hamiltonian constraint (21) is

$$-i\{Y, V(x)\} + \{Q, V(x)\} = -\left(\frac{G\Lambda}{2}\right)V(x). \quad (23)$$

We will now make use of a remarkable identity

$$\{-iI_{CS}[A], V(x)\} = \{\text{Im}[I_{CS}[A]], V(x)\}, \quad (24)$$

namely that the Poisson bracket of $V(x)$ with the Chern-Simons functional $I_{CS}[A]$ is the same as the Poisson bracket of $V(x)$ with its imaginary part. This is proven in Appendix A. This means that the first term on the left hand side of (23) is zero, which implies the following fundamental Poisson brackets

$$\{Q, V(x)\} = -\left(\frac{G\Lambda}{2}\right)V(x); \quad \{Y, V(x)\} = 0. \quad (25)$$

Of note is that the imaginary part Q of the Chern-Simons functional I_{CS} admits an affine Lie algebraic structure with $V(x)$ for each x , while the real part Y acts as a Casimir function. We will regard the latter as proportional to the identity operator in the quantum theory.

We will now return to a point which we postponed from (19) vis-a-vis (13). Noting that $Q = \frac{1}{2i}(I_{CS} - I_{CS}^\dagger) = Q^\dagger$, and $V^\dagger = V$; it follows from Eq.(14) that (in the quantum operators we omit the hats for simplicity, and they will be restored when necessary)

$$[Q, V^2(x)] = -2i\lambda V^2(x); \quad \lambda = \frac{\hbar G\Lambda}{2}. \quad (26)$$

In order to formulate the Hamiltonian constraint as a weight one scalar density and to make contact with the affine representation, we postulate

$$[Q, V(x)] = -i\lambda V(x) \quad (27)$$

as the equation of the Hamiltonian constraint defining physical states. It holds at classical Poisson bracket level, and it is also a sufficient condition (albeit not necessary) for Eqs.(14) and (26) [20].

Algebraic quantization

We will now proceed with the quantization of the theory, bringing in the relevant steps from the algebraic quantization program [7], [16].

(i) Step 1: Our first step will be to identify an appropriate set S of gauge-invariant classical observables, which is closed under Poisson brackets and under complex conjugation

$$S = \{I, Q, V(x)\}_{\forall x \in \Sigma}. \quad (28)$$

Encoded in the requirement that the set S be closed under Poisson brackets is the solution to the local Hamiltonian constraint (25). Combined with the fact that S is invariant under the identity-connected component of complex $SO(3)$ transformations, this will address the Gauss' law and Hamiltonian constraints. We will address the diffeomorphism constraint shortly.

(ii) Step 2: Each function in S will be regarded as an elementary classical variable which is to have an unambiguous quantum analogue. Since Q and $V(x)$, taken to be the basic variables, are composite objects constructed purely from

coordinate $A_i^a(x)$ and momentum $\tilde{E}_a^i(x)$ variables respectively from the original Ashtekar phase space, then their quantum analogues will be free of ordering ambiguities.

With each element $I, Q, V(x) \in S$ we associate an abstract operator $\hat{I}, \hat{Q}, \hat{V}(x)$, which defines the free associative algebra B_{aux} generated by these elementary quantum operators. Upon this we impose the canonical commutation relation, consistent with the Heisenberg–Dirac promotion of classical Poisson brackets to quantum commutators

$$\{Q, V(x)\} = -\left(\frac{G\Lambda}{2}\right)V(x) \longrightarrow \frac{1}{(i\hbar)}[\hat{Q}, \hat{V}(x)] = -\left(\frac{G\Lambda}{2}\right)\hat{V}(x). \quad (29)$$

Then using the definitions (22), the quantum Hamiltonian constraint can be written as an infinite number of affine commutation relations, one affine commutation relation per spatial point x

$$[\hat{Q}, \hat{V}(x)] = -i\lambda\hat{V}(x); \quad [\hat{Q}, \hat{Q}] = [\hat{V}(x), \hat{V}(y)] = 0. \quad (30)$$

From the commutation relations (30) can be immediately written down the following exponentiated point-wise relations for any real numbers a and b

$$e^{-ia\hat{Q}}\hat{V}(x)e^{ia\hat{Q}} = e^{-\lambda a}\hat{V}(x); \quad e^{-ib\hat{V}(x)}\hat{Q}e^{ib\hat{V}(x)} = \hat{Q} + \lambda b\hat{V}(x). \quad (31)$$

Since $V(x)$ is a locally defined object of density weight one, it is not diffeomorphism invariant. It transforms under spatial diffeomorphisms parametrized by any smooth vector field $\xi^i \in C^\infty(\Sigma)$ as

$$\delta_\xi V(x) = \mathbf{L}_\xi V(x) = \partial_i(\xi^i(x)V(x)) \neq 0, \quad (32)$$

which means that the local Hamiltonian constraint $H(x) = 0$ is also not diffeomorphism invariant. But we want our states to be solutions of the diffeomorphism constraint $H_i(x) = 0$ in addition to the Gauss’ law constraint $G_a(x) = 0$, while at the same time lying in the kernel of $H(x)$. That is, we want *physical* states, or elements of the physical Hilbert space \mathbf{H}_{Phys} . To this end it will be apposite to construct a *global* total volume functional V from the *local* function $V(x)$, given by

$$V = \int_\Sigma d^3x V(x). \quad (33)$$

Note that V , the volume of 3-space Σ , is a diffeomorphism-invariant quantity. The following exponentiated relations, a weaker form of (31) in relation to the Hamiltonian constraint $H(x)$, are also true:

$$e^{-ia\hat{Q}}\hat{V}e^{ia\hat{Q}} = e^{-\lambda a}\hat{V}; \quad e^{-ib\hat{V}}\hat{Q}e^{ib\hat{V}} = \hat{Q} + \lambda b\hat{V}. \quad (34)$$

It is important to maintain the distinction between the first equation of (31), which uses the local $\hat{V}(x)$, and (34), which uses the integrated form \hat{V} . The former provides the *local* diffeomorphism non-invariant Hamiltonian constraint, while the latter with provide diffeomorphism-invariant states satisfying this constraint.

(iii) Step 3: Next, we introduce an involution operation $*$ on B_{aux} , which defines an algebra B_{aux}^* . So one must have $(A^*)^* = A$, $(B^*)^* = B$, and $(AB)^* = A^*B^*$ and similarly for the products in opposite order. The quantum analogues of A and B must be self-adjoint operators such that

$$A^* \longrightarrow \hat{A}^\dagger; \quad B^* \longrightarrow \hat{B}^\dagger; \quad (\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \quad (35)$$

for all A, B . Applying this to (30) one sees that

$$[\hat{Q}^\dagger, \hat{V}^\dagger(x)] = -i\lambda\hat{V}^\dagger(x); \quad [\hat{Q}^\dagger, \hat{Q}^\dagger] = [\hat{V}^\dagger(x), \hat{V}^\dagger(y)] = 0. \quad (36)$$

The result is that the set of elementary observables S is not only closed under Poisson brackets and complex conjugation, but also is consistent with the involution operation $*$. The existence of self-adjoint operators \hat{Q} and \hat{V} requires that a physical Hilbert space \mathbf{H}_{Phys} be defined, such that the \dagger operation acts by Hermitian conjugation. We use the term ‘physical’ since all of the constraints will have inherently been implemented.

(iv) Step 4: Ultimately we will construct a linear $*$ -representation π of the abstract algebra $B_{aux}^{(*)}$ via linear operators on the physical Hilbert space \mathbf{H}_{Phys} . For the remaining steps of the quantization procedure we will proceed along a different path to the one presented in [16], as we will now be addressing the quantization of the Hamiltonian constraint and the physical states from a different interpretation.

On the one hand it could be argued that S might not be ‘large enough’ since it essentially consists merely of three elements. After all, there are an infinite number of sufficiently regular functions on the original Ashtekar phase space $(A_i^a, \tilde{E}_a^i, I)$ which cannot be obtained as a sum of products of elements of S . But on the other hand, the results of this paper will show that S is ‘large enough’ to admit a nontrivial, unitary representation of its associated Lie algebra on a Hilbert space.

Construction of the physical Hilbert space

Having put in place the necessary elements, we will now proceed with the construction of a physical Hilbert space \mathbf{H}_{Phys} . Let us define for our fiducial vector $|\eta\rangle = |0, 0\rangle$, a gauge-invariant, diffeomorphism-invariant state lying in the kernel of the quantum constraints $|0, 0\rangle \in Ker\hat{C}$, where $\hat{C} = \{\hat{G}_a(x), \hat{H}_i(x), \hat{H}(x)\}$ are the Gauss' law, diffeomorphism and Hamiltonian constraints. By $Ker\{\hat{H}(x)\}$, we mean that the Hamiltonian constraint acting on $|\eta\rangle$ is given by the affine commutation relation

$$\lambda\hat{V}(x)|0, 0\rangle = [i\hat{Q}, \hat{V}(x)]|0, 0\rangle, \quad (37)$$

which is a restatement of $\hat{H}(x) = 0$. In this interpretation, $|0, 0\rangle$ plays the role of a sort of "ground state". Now act on both sides of (37) with the unitary operator $e^{-ia\hat{Q}}$ where a is an arbitrary, real-valued dimensionless numerical constant, which gives

$$\begin{aligned} \lambda e^{-ia\hat{Q}}\hat{V}(x)|0, 0\rangle &= \lambda(e^{-ia\hat{Q}}\hat{V}(x)e^{ia\hat{Q}})e^{-ia\hat{Q}}|0, 0\rangle \\ &= [e^{-ia\hat{Q}}\hat{V}(x)e^{ia\hat{Q}}, -ie^{-ia\hat{Q}}\hat{Q}e^{ia\hat{Q}}]e^{-ia\hat{Q}}|0, 0\rangle. \end{aligned} \quad (38)$$

Using the first equation of (31) as well as $[\hat{Q}, \hat{Q}] = 0$ from (30), equation (38) simplifies to

$$\lambda e^{-\lambda a}\hat{V}(x)e^{-ia\hat{Q}}|0, 0\rangle = e^{-\lambda a}[\hat{V}(x), -i\hat{Q}]e^{-ia\hat{Q}}|0, 0\rangle. \quad (39)$$

Cancelling off the induced dilation pre-factor $e^{\lambda a}$ and defining a dilated state

$$|a, 0\rangle = e^{-ia\hat{Q}}|0, 0\rangle, \quad (40)$$

then (39) can be written as

$$\lambda\hat{V}(x)|a, 0\rangle = [\hat{V}(x), -i\hat{Q}]|a, 0\rangle, \quad (41)$$

which is none other than the local Hamiltonian constraint $\hat{H}(x) = 0$ acting on the dilated state $|a, 0\rangle$. So given that $|0, 0\rangle \in Ker\hat{C}$, it follows that $|a, 0\rangle \in Ker\hat{C}$ as well.

Next, act on both sides of (37) with the unitary operator $e^{-ib\hat{V}}$ and define the translated state, translated in the carrier space of the affine group

$$|0, b\rangle = e^{-ib\hat{V}}|0, 0\rangle, \quad (42)$$

where b is an arbitrary, real-valued numerical constant of mass dimension $[b] = 3$. This yields

$$\begin{aligned} \lambda e^{-ib\hat{V}}\hat{V}(x)e^{ib\hat{V}}e^{-ib\hat{V}}|0, 0\rangle \\ = [e^{-ib\hat{V}}\hat{V}(x)e^{ib\hat{V}}, -ie^{-ib\hat{V}}\hat{Q}e^{ib\hat{V}}]e^{-ib\hat{V}}|0, 0\rangle. \end{aligned} \quad (43)$$

Using the second equation of (34) to the left of the comma in the commutator of (43) and $[\hat{V}, \hat{V}(x)] = 0$ to the right of the comma (note that the latter manipulation is simply the fact that the integrated form of the volume operator and the local form must commute. This can be seen from summing the last equality of (30) over all $y \in \Sigma$), this yields

$$\lambda\hat{V}(x)|0, b\rangle = [\hat{V}(x), -i\hat{Q}]|0, b\rangle - i\lambda b[\hat{V}, \hat{V}(x)]|0, b\rangle. \quad (44)$$

As explained in the previous footnote, the second term on the right hand side of (44) is zero, which leaves us with

$$\lambda\hat{V}(x)|0, b\rangle = [\hat{V}(x), -i\hat{Q}]|0, b\rangle, \quad (45)$$

which is none other than the local Hamiltonian constraint $\hat{H}(x) = 0$ acting on the translated state $|0, b\rangle$. So given that $|0, 0\rangle \in Ker\hat{C}$, it follows that $|0, b\rangle \in Ker\hat{C}$ as well.

General quantum Affine group element

Having illustrated the idea for physical states corresponding to transformations by a and by b individually, we will now consider the two transformations applied together. Define the general affine coherent state

$$|a, b\rangle = U[a, b]|0, 0\rangle = e^{-ia\hat{Q}}e^{-ib\hat{V}}|0, 0\rangle, \quad (46)$$

and let us act on both sides of the local Hamiltonian constraint (37) with an arbitrary group element $U[a, b]$. This yields

$$\begin{aligned} & \lambda e^{-ia\hat{Q}}e^{-ib\hat{V}}\hat{V}(x)e^{ib\hat{V}}e^{ia\hat{Q}}(e^{-ia\hat{Q}}e^{-ib\hat{V}}|0, 0\rangle) \\ &= \left[e^{-ia\hat{Q}}e^{-ib\hat{V}}\hat{V}(x)e^{ib\hat{V}}e^{ia\hat{Q}}, -ie^{-ia\hat{Q}}e^{-ib\hat{V}}\hat{Q}e^{ib\hat{V}}e^{ia\hat{Q}}, \right] e^{-ia\hat{Q}}e^{-ib\hat{V}}|0, 0\rangle. \end{aligned} \quad (47)$$

Using $[\hat{V}, \hat{V}(x)] = [\hat{Q}, \hat{Q}] = 0$ as well as the second equation of (34), then equation (47) simplifies to

$$\lambda e^{-ia\hat{Q}}\hat{V}(x)e^{ia\hat{Q}}|a, b\rangle = \left[ie^{-ia\hat{Q}}(\hat{Q} + \lambda b\hat{V})e^{ia\hat{Q}}, e^{-ia\hat{Q}}\hat{V}(x)e^{ia\hat{Q}} \right] |a, b\rangle. \quad (48)$$

Proceeding along, using the relations, this further simplifies to

$$e^{-\lambda a}\lambda\hat{V}(x)|a, b\rangle = \left[i\hat{Q} + i\lambda b e^{-\lambda a}\hat{V}, e^{-\lambda a}\hat{V}(x) \right] |a, b\rangle. \quad (49)$$

The second commutator on the right hand side of (49) vanishes on account of $[\hat{V}, \hat{V}(x)] = 0$. Cancelling off the common dilation pre-factor $e^{\lambda a}$, we have

$$\lambda\hat{V}(x)|a, b\rangle = i[\hat{Q}, \hat{V}(x)]|a, b\rangle \quad (50)$$

which is simply the local Hamiltonian constraint $\hat{H}(x) = 0$ acting on the general coherent state $|a, b\rangle$. The result is that given any fiducial state $|0, 0\rangle \in \text{Ker}\hat{H}_i(x), \hat{G}_a(x)$, it follows that an arbitrary coherent state $|a, b\rangle = U[a, b] \in \text{Ker}\hat{C}$ lies in the kernel of all of the constraints.

Normalizability and inner product

If a fiducial state $|0, 0\rangle$ satisfies the admissibility condition (58), then all states $|a, b\rangle$ are unitarily related to $|0, 0\rangle$, and form an overcomplete basis of physical coherent states. These results and the results which follow are actually independent of the specific carrier representation space of the affine group. Let the fiducial state be normalized such that

$$\langle 0, 0|0, 0\rangle = 1. \quad (51)$$

We would like to find the inner product between two states $|a, b\rangle$ and $|a', b'\rangle$. This is given by

$$\begin{aligned} \langle a', b'|a, b\rangle &= \langle 0, 0|e^{ib'\hat{V}}e^{ia'\hat{Q}}e^{-ia\hat{Q}}e^{-ib\hat{V}}|0, 0\rangle \\ &= \langle 0, 0|e^{ib'\hat{V}}e^{i(a'-a)\hat{Q}}e^{-ib\hat{V}}|0, 0\rangle. \end{aligned} \quad (52)$$

Proceeding from (52) and using the trick $I = e^{-\hat{A}}e^{\hat{A}}$, we have

$$\begin{aligned} \langle a', b'|a, b\rangle &= \langle 0, 0|e^{i(a'-a)\hat{Q}}e^{-i(a'-a)\hat{Q}}e^{ib'\hat{V}}e^{i(a'-a)\hat{Q}}e^{-ib\hat{V}}|0, 0\rangle \\ &= \langle 0, 0|e^{i(a'-a)\hat{Q}}\exp\left[ib'e^{-\lambda(a'-a)}\hat{V}\right]e^{-ib\hat{V}}|0, 0\rangle \\ &= \langle 0, 0|e^{i(a'-a)\hat{Q}}\exp\left[i(-b+b'e^{-\lambda(a'-a)})\hat{V}\right]|0, 0\rangle \\ &= \langle 0, 0|a - a', b - b'e^{\lambda(a-a')}\rangle. \end{aligned} \quad (53)$$

As (53) shows, the overlap between two states is equivalent to performing two affine group transformations in opposite directions. A cancellation of the group action to the identity element occurs only for $a = a'$ and $b = b'$.

SUMMARY AND DISCUSSION

We have provided a quantization of gravity for spacetimes of Lorentzian signature with nonzero cosmological constant. Using the concept of the affine quantization program in conjunction with the relevant steps of the algebraic quantization procedure, we have constructed a physical Hilbert space of what could be classified as gravitational coherent states. The group theoretical aspect of quantization enables the infusion of results and techniques from wavelet transform theory into quantum gravity, where the powerful coherent state machinery is available. In this approach one can dispense with direct reference to the quantum operators $\widehat{V}(x)$, \widehat{V} and \widehat{Q} , and think of the carrier space in terms of coherent states encoding the group action. The results of this paper depend only on the existence of fiducial vectors satisfying the admissibility condition, and are independent of the specific representation or polarization of the carrier space.

Some future directions of research will include (i) application to the Chern-Simons state, (ii) the possible enlargement of the set S to include more general objects, (iii) the computation of expectation values and observables, and (iv) the incorporation of matter fields into the formalism.

APPENDIX A. MEASURE, INNER PRODUCT, AND COHERENT STATES ASSOCIATED WITH THE AFFINE GROUP

The action of the affine group G_{aff} on the real numbers has the following matrix representation

$$U[a, b] \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

which has a natural action on itself by left-matrix multiplication via the group multiplication law

$$U[a, b]U[a', b'] = U[aa', b + ab']. \quad (54)$$

The affine group has invariant left Haar and right Haar measures $d\mu_l$ and $d\mu_r$ respectively, given by

$$d\mu_l[a, b] = \frac{db \wedge da}{a^2}; \quad d\mu_r[a, b] = \frac{db \wedge da}{a}. \quad (55)$$

While these two measures are equivalent in the sense of measure theory, they are not the same measure. Hence the affine group is not a unimodular group. With respect to the unitary representation of the group one sees, from exponentiation of (20), that the parametrization of the general group element as the linear operator

$$U[a, b] = e^{-ibq} e^{iD \ln a} \quad (56)$$

correctly reproduces the group multiplication law (54). Using any normalized fiducial vector $|\eta\rangle$ with $\langle\eta|\eta\rangle = 1$, one can define an overcomplete basis of unit vectors as

$$|a, b\rangle = U[a, b]|\eta\rangle. \quad (57)$$

So for each fiducial vector $|\eta\rangle$, there exists a set of states labelled by a and b . Since there is a large class of possible fiducial vectors $|\eta\rangle$, then one has certain freedom in the choice of Hilbert spaces (one could go further, utilizing the admissibility condition for fiducial vectors in order to construct reproducing kernel Hilbert spaces (See e.g. [14])).

Equation (57) bears an analogy to continuous wavelet transform theory [17], where $|a, b\rangle$, a set of coherent states, plays the role of the wavelet transform of a mother wavelet (signal) $|\eta\rangle$. A necessary condition is that $|\eta\rangle$ satisfy a certain admissibility condition predicated on its existence as an element of the set $\psi \in L^2(R, dz)$. The admissibility condition is

$$c_\psi = 2\pi \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|} |\phi(\xi)|^2 < \infty, \quad (58)$$

where ϕ is the Fourier transform of ψ . Note that the unitary action of (56) on $|\eta\rangle$ translates in the language of functions into

$$\psi'(z) = U[a, b]\psi = |a|^{-1/2} \psi\left(\frac{z-b}{a}\right), \quad (59)$$

where $b \in R$ and $a \neq 0$. That $U[a, b]$ is unitary is evident in the fact that it preserves the Hilbert space norm

$$\|\psi\|^2 = \int_{-\infty}^{\infty} dz |\psi'(z)|^2. \quad (60)$$

In the context of quantum gravity, the square integrability of the representation $U[a, b]$ implies the existence of fiducial vectors ψ for which the matrix element $\langle U[a, b]\psi | \psi \rangle$ is square integrable as a function of the labels a and b with respect to the left Haar measure

$$\int_{G_{aff}} d\mu_l[a, b] |\langle U[a, b]\psi | \psi \rangle|^2 < \infty. \quad (61)$$

Additionally, since the Fourier transform is a linear isometry, it follows that the Fourier transformed version of (56) also provides a unitary representation of the affine group in its action on ϕ . Both representations are irreducible.

In this paper we will use the ‘affine conjugate’ pair $(\widehat{V}, \widehat{Q})$ where V is the integral of $V(x)$ over all space (the volume operator), to describe quantum gravity. Our application is relevant mostly as an alternative to LQG and is somewhat different from that described by the affine conjugate pair $(\widehat{q}_{ij}, \widehat{\pi}_i^k)$ due to Klauder, namely the spatial metric and the field $\widehat{\pi}_i^k$ related to the conjugate momentum $\widehat{\pi}^{kl}$ of the spatial metric. Closure of the constraints algebra in canonical quantum gravity described by $(\widehat{q}_{ij}, \widehat{\pi}^{kl})$ produces second-class constraints [18]. This is unlike the case as described in the Ashtekar variables (A_i^a, \widehat{E}_a^i) , where the constraints algebra is first-class. Hence for this paper, we will use the affine commutator of \widehat{Q} and \widehat{V} to represent the symmetric ordering of the Hamiltonian constraint, which at the same time addresses the issues mentioned regarding LQG.

In the following, we want to construct the coherent state framework according to the article [14] by using the affine conjugate pair $(\widehat{Q}, \widehat{V})$, that is another application for the affine representation. We also can use the spatial metric matrix field g and the momentum matrix field π to describe this topic. Articles [13], [18] provide further descriptions. Here, the commutation relation for $(\widehat{V}, \widehat{Q})$ is suitable for the one-dimension affine algebra: From the formula (for ease of discussion here we set $\lambda = 1$):

$$\widehat{Q} = \frac{1}{2}[\widehat{\theta}\widehat{V} + \widehat{V}\widehat{\theta}] \quad , \quad \widehat{\theta} = -i\frac{\partial}{\partial\widehat{V}} \quad (62)$$

which is also equal to:

$$\widehat{Q} = \frac{1}{2}[-i\frac{\partial}{\partial\widehat{V}}\widehat{V} + \widehat{V}(-i\frac{\partial}{\partial\widehat{V}})] = -\frac{i}{2} - i\widehat{V}\frac{\partial}{\partial\widehat{V}} \quad (63)$$

With the inner product

$$\langle \phi | \psi \rangle = \int_0^\infty \phi(\widehat{V})^* \psi(\widehat{V}) d\widehat{V}$$

The operator \widehat{Q} generates the unitary dilations in the representation space:

$$e^{-ia\widehat{Q}}\psi(\widehat{V}) = e^{-\frac{a}{2}}\psi(e^{-a}\widehat{V})$$

$$\|e^{-ia\widehat{Q}}|\psi\rangle\| = \| |\psi\rangle \|$$

And from the definition for unitary operators $U(b, a) = e^{ib\widehat{V}}e^{-ia\widehat{Q}}$ with the composition rule $U(b', a')U(b, a) = U(b' + e^{-a'}b, a' + a)$.

We can construct a set of coherent states:

$$|b, a\rangle = U(b, a)|\eta\rangle$$

where $|\eta\rangle$ is an unspecified normalized fiducial vector in the representation space. The family of coherent states provides a resolution of unity in the form:

$$N^{-1} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} da e^a |b, a\rangle \langle b, a| = \mathbb{I},$$

where $N = 2\pi \int_0^\infty d\widehat{V} \widehat{V}^{-1} |\eta(\widehat{V})|^2 < \infty$, is finite.

For minimum uncertainty states, one takes the Heisenberg uncertainty principle into consideration. In general quantum mechanics, any two observable operators \hat{A} , \hat{B} are consistent with the uncertainty relation:

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^2$$

But on the affine representation space, any two self-adjoint operators can be consistent with the general affine uncertainty relation: (here, the two operators are \hat{V} and \hat{Q}):

$$\langle\Delta\hat{V}\rangle^2\langle\Delta\hat{Q}\rangle^2 \geq \frac{\langle\hat{V}\rangle^2}{4}$$

We can use the general analysis just like that in [14] to prove the affine uncertainty relation by taking a normalized fiducial vector with of form $\eta_1(\hat{V}) = C_1(\alpha', \beta')\hat{V}^{\alpha'}e^{-\beta'\hat{V}}$. Where α', β' are positive real coefficients, $C_1(\alpha', \beta')$ is the normalization constant.

APPENDIX B. PROOF THAT THE REAL PART OF THE CHERN-SIMONS FUNCTIONAL POISSON-COMMUTES WITH THE VOLUME ELEMENT

Consider the Chern-Simons functional: $I_{CS}[A] = \frac{1}{2}\int_{\Sigma}[A^a \wedge dA_a + \frac{1}{3}\epsilon^{abc}A_a \wedge A_b \wedge A_c]$. Using the definition (1), and defining the one form $k^a = K_i^a dx^i$, expansion of the Chern-Simons functional in k and Γ leads straightforwardly to the expression

$$I_{CS}[A] = I_{CS}[\Gamma] + \gamma \int_M R_a^\Gamma \wedge k^a + \frac{\gamma^2}{2!} \int_M k^a \wedge (D^\Gamma k)_a + \frac{\gamma^3}{3!} \int_M \epsilon^{abc} k_a \wedge k_b \wedge k_c,$$

where $R_a^\Gamma = d\Gamma_a + \frac{1}{2}\epsilon_a^{bc}\Gamma_b \wedge \Gamma_c$ is the curvature two form of the connection one form Γ^a . Consider the Poisson bracket of the volume functional and the Chern-Simons functional (for brevity we suppress the label of spatial points x . The results which follow equally apply to the global volume functional V , just as they apply to the local $V(x)$. Note also that $\{F, \sqrt{\det \tilde{E}}\} = \frac{1}{2\sqrt{\det \tilde{E}}}\{F, \det \tilde{E}\}$ for all F). To wit,

$$\begin{aligned} \{V, I_{CS}[A]\} &= \{V, I_{CS}[\Gamma]\} + \gamma \int_M \{V, R_a^\Gamma \wedge k^a\} + \frac{\gamma^2}{2!} \int_M \{V, k^a \wedge (D^\Gamma k)_a\} + \frac{\gamma^3}{3!} \int_M \{V, \epsilon^{abc} k_a \wedge k_b \wedge k_c\} \\ &= \{V, I_{CS}[\Gamma]\} + \frac{\gamma^2}{2!} \int_M k^a \wedge (D^\Gamma k)_a + \{V, \gamma \int_M R_a^\Gamma \wedge k^a + \frac{\gamma^3}{3!} \int_M \epsilon^{abc} k_a \wedge k_b \wedge k_c\} \\ &= \{V, \text{Re}[(I_{CS}[A])]\} + \{V, \text{Im}(I_{CS}[A])\} \quad (\gamma = \pm i) \\ &= \{V, \text{Im}[I_{CS}[A]]\}. \end{aligned}$$

The result can be explained by the following observations:

- (1) The term $\{V, I_{CS}[\Gamma]\}$ is zero because Γ is a function only of \tilde{E} , thus Poisson-commuting with V .
- (2) The term $\frac{\gamma^2}{2!} \int_M \{V, k^a \wedge (D^\Gamma k)_a\}$ also vanishes. We note that

$$\begin{aligned} \frac{\gamma^2}{2!} \int_M \{V, k^a \wedge (D^\Gamma k)_a\} &\propto \{\epsilon_{lmn}\epsilon_{dbc}\tilde{E}^{ld}\tilde{E}^{mb}\tilde{E}^{nc}, \int \epsilon^{ijk'} k_{k'a} D_i^\Gamma k_j^a\} \\ &= \int \epsilon_{lmn}\epsilon_{dbc}\epsilon^{ijk'} \{\tilde{E}^{ld}\tilde{E}^{mb}\tilde{E}^{nc}, k_{k'a} D_i^\Gamma k_j^a\} \\ &= \int \epsilon_{lmn}\epsilon_{dbc}\epsilon^{ijk'} (\{\tilde{E}^{ld}, k_{k'a} D_i^\Gamma k_j^a\} \tilde{E}^{mb}\tilde{E}^{nc} + \tilde{E}^{ld}\{\tilde{E}^{mb}, k_{k'a} D_i^\Gamma k_j^a\} \tilde{E}^{nc} \\ &\quad + \tilde{E}^{ld}\tilde{E}^{mb}\{\tilde{E}^{nc}, k_{k'a} D_i^\Gamma k_j^a\}). \end{aligned} \tag{64}$$

We now focus on the Poisson bracket $\{\tilde{E}^{ld}, k_{k'a} D_i^\Gamma k_j^a\}$, namely

$$\{\tilde{E}^{ld}, k_{k'a} D_i^\Gamma k_j^a\} = \{\tilde{E}^{ld}, k_{k'a}\} D_i^\Gamma k_j^a + k_{k'a} \{\tilde{E}^{ld}, D_i^\Gamma k_j^a\}.$$

Note that

$$\{\tilde{E}^{ld}, D_i^\Gamma k_j^a\} = D_i^\Gamma \{\tilde{E}^{ld}, k_j^a\} \propto \delta_j^l \delta_a^d (D_i^\Gamma \delta^3)$$

where we have used the shorthand notation $\delta^3 \equiv \delta^{(3)}(x, y)$. From this Poisson bracket $\{\tilde{E}^{ld}, D_i^\Gamma k_j^a\} = \tilde{E}^{ld}(D_i^\Gamma k_j^a) - (D_i^\Gamma k_j^a)\tilde{E}^{ld} \propto \delta_j^l \delta_a^d (D_i^\Gamma \delta^3)$, $\implies (D_i^\Gamma k_j^a)\tilde{E}^{ld} = \tilde{E}^{ld}(D_i^\Gamma k_j^a) - \delta_j^l \delta_a^d (D_i^\Gamma \delta^3)$. So we have

$$\begin{aligned} \{\tilde{E}^{ld}, k_{k'a} D_i^\Gamma k_j^a\} &= \{\tilde{E}^{ld}, k_{k'a}\} D_i^\Gamma k_j^a + k_{k'a} \{\tilde{E}^{ld}, D_i^\Gamma k_j^a\} \\ &= \{\tilde{E}^{ld}, k_{k'a}\} D_i^\Gamma k_j^a + k_{k'a} D_i^\Gamma \{\tilde{E}^{ld}, k_j^a\} \propto \delta_{k'}^l \delta_a^d \delta^3 (D_i^\Gamma k_j^a) + k_{k'a} \delta_j^l \delta_a^d (D_i^\Gamma \delta^3) \end{aligned}$$

The three terms in equation (64) will each yield the same result, which can be seen by a relabelling of indices. So it suffices to illustrate the calculation for one term. In this process, we will also use $\{\tilde{E}^{ia}, k_{jb}\} \propto \delta_j^i \delta_b^a \delta^3$.

Consider the term $\int \epsilon_{lmn} \epsilon_{dbc} \epsilon^{ijk'} \{\tilde{E}^{ld}, k_{k'a} D_i^\Gamma k_j^a\} \tilde{E}^{mb} \tilde{E}^{nc}$ in equation (64):

$$\begin{aligned} & \int \epsilon_{lmn} \epsilon_{dbc} \epsilon^{ijk'} \{\tilde{E}^{ld}, k_{k'a} D_i^\Gamma k_j^a\} \tilde{E}^{mb} \tilde{E}^{nc} \\ & \propto \int \epsilon_{lmn} \epsilon_{dbc} \epsilon^{ijk'} \delta_{k'}^l \delta_a^d \delta^3 (D_i^\Gamma k_j^a) \tilde{E}^{mb} \tilde{E}^{nc} + \int \epsilon_{lmn} \epsilon_{dbc} \epsilon^{ijk'} k_{k'a} \delta_j^l \delta_a^d (D_i^\Gamma \delta^3) \tilde{E}^{mb} \tilde{E}^{nc} \\ & \propto \int \epsilon^{lij} \delta^3 (D_i^\Gamma k_{ja}) \tilde{E}_l^a + \int (D_i^\Gamma \delta^3) \epsilon^{ijk'} k_{k'a} \tilde{E}_j^a \\ & = - \int \delta^3 D_i^\Gamma (\epsilon^{ilj} \tilde{E}_l^a k_{ja}) + \int (D_i^\Gamma \delta^3) \epsilon^{ijk'} \tilde{E}_j^a k_{k'a} \\ & \propto - \int \delta^3 D_i^\Gamma [\epsilon_{da'b'} \tilde{E}^{ja'} \tilde{E}^{ib'} k_j^d] + \int (D_i^\Gamma \delta^3) [\epsilon_{da'b'} \tilde{E}^{k'a'} \tilde{E}^{ib'} k_{k'}^d] \\ & = \int \delta^3 D_i^\Gamma [\tilde{E}^{ia} (\epsilon_{a'da} \tilde{E}^{ja'} k_j^d)] - \int (D_i^\Gamma \delta^3) [\tilde{E}^{ia} (\epsilon_{a'da} \tilde{E}^{k'a'} k_{k'}^d)] \\ & = \int \delta^3 D_i^\Gamma [\tilde{E}^{ia} G_a] - \int (D_i^\Gamma \delta^3) [\tilde{E}^{ia} G_a] = 2 \int \delta^3 D_i^\Gamma [\tilde{E}^{ia} G_a] = 0 \end{aligned}$$

wherein $G_a = \epsilon_{aa'd} \tilde{E}^{ja'} k_j^d$ is just the Gauss Law constraint in the ADM triad formulation of gravity. So equation (64) provides no contribution and $\frac{\gamma^2}{2!} \int_M \{\hat{V}, k^a \wedge (D^\Gamma k)_a\} = 0$ as desired.

(3) Therefore, the result is that $\{\hat{V}, Im(I_{CS}[A])\} = \{\hat{V}, I_{CS}[A]\}$ since $\{\hat{V}, Re(I_{CS}[A])\} = 0$.

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- [19] There is a class of solutions based on holonomies of analytical closed loops, the so-called loop states. These states correspond to degenerate geometries, whose classical interpretation remains unclear.
- [20] For the actual construction of states in this paper we will also need the integrated form of $V(x)$ and not of $V^2(x)$. This is because since the former is diffeomorphism invariant in the field theoretical sense in addition to being ultraviolet self-regulating whereas the latter, while congruous with a symmetric Weyl factor ordering, is not. It can be shown that the physical Hilbert space \mathbf{H}_{Phys} , which we will later construct, is invariant under the replacements $\hat{V}(x) \rightarrow \hat{V}^n(x)$ and $\lambda \rightarrow n\lambda$ in the quantum Hamiltonian constraint, whence (26) and (27) are special cases corresponding to $n = 2$ and $n = 1$ respectively. The weight one ($n = 1$) case is superior for the reasons that we have outlined, and therefore will constitute the basis for the results of this paper.